

MORE REDUCED OBSTRUCTION THEORIES

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ABSTRACT. We first develop a general formalism for globally removing factors from an obstruction theory. We then apply this formalism to give a construction of a reduced obstruction theory on the moduli space of morphisms from a curve to a surface $f: C \rightarrow S$ in class β such that $H^1(C, f^*T_S) \xrightarrow{-\cup\beta} H^2(S, \mathcal{O}_S)$ is surjective. This condition appears in recent work of Kool and Thomas.

1. INTRODUCTION

Due to the deformation invariance of Gromov-Witten invariants, smooth complex projective surfaces having deformations in a direction where the topological class $\beta \in H^{1,1}(S, \mathbb{C}) \cap H^2(S, \mathbb{Z})$ does not stay of type $(1, 1)$ have no Gromov-Witten invariants.

This can be fixed by introducing a *reduced perfect obstruction theory* for the moduli of stable maps. This obstruction theory is obtained by removing a factor from the usual obstruction theory, and is only invariant under deformations for which β stays of the given type. This technique has been used extensively in the case of $K3$ -surfaces by Maulik–Pandharipande [13] and by Maulik–Pandharipande–Thomas [14]. It has proven difficult though to show that the complex obtained by removing a factor from the standard obstruction theory indeed satisfies all requirements for being an obstruction theory. This is closely related to the problem of showing that realized obstructions lie in the kernel of a semi-regularity map. To treat other kinds of surfaces as well, an alternative approach has been introduced recently by Kool and Thomas [9]. There a complex having all formal properties expected from the reduced obstruction theory is defined using an *algebraic twistor family*. The aim of this note is to construct a reduced obstruction theory in these cases directly without resorting to an algebraic twistor family.

The basic idea how to remove a factor from an obstruction theory presented here is by mapping the moduli problem in question to a further smooth moduli problem, which nonetheless has a non-trivial obstruction theory. In the case of moduli of sheaves on a surface this was already used by Mukai [15] and Artamkin [1]. The difficult part is to ensure that the obstruction theories of the two moduli problems are compatible. In the context of formal moduli problems, it was observed by Manetti [12] that this is automatic as soon as the morphism of formal moduli problems is induced by a morphism of differential graded Lie algebras. This technique allows us to show that obstructions to a formal moduli problem lie in the kernel of the induced map of the obstruction spaces.

Knowing the compatibilities of the obstruction theories of the formal moduli problems corresponding to closed points of the global moduli space is not

enough though to globally remove a factor. As an example for the amount of calculations necessary, see the example of Donaldson–Thomas invariants [18, Section 3], or for the case of Pandharipande–Thomas invariants [6]. In both cases the obstructions have to be considered with respect to a fixed determinant. It requires compatibilities of the obstructions not only over square-zero extensions of Artinian rings, but over square-zero extensions of arbitrary bases.

In this note we study the case where a strong compatibility of the obstruction theories is available globally on the moduli space. As compatibility datum between the obstruction theories we require more than just commutativity in the derived category. Instead, we assume that the diagram of obstruction theories commutes up to homotopy in some higher categorical model. We show that in this case a factor of the obstruction theory can be removed globally on the entire moduli space in question. Using commutativity up to homotopy instead of commutativity in the derived category and cofiber sequences instead of exact triangles makes the necessary calculations a breeze.

This raises the question where such strong compatibilities can be found. Natural examples where such compatibilities up to homotopy are available come from derived algebraic geometry. In a previous work [17], a morphism from the derived moduli space of stable maps to the derived Picard stack was introduced. Assuming the compatibility of the obstruction theories obtained from this morphism, we show that a factor of the obstruction theory can be removed globally.

Conventions. We have tried to adhere to the following conventions. We work throughout over an arbitrary base ring k , which in Section 4 becomes the field of complex numbers. We will denote the cotangent complex of a scheme, or more generally an Artin stack, over k by L_X instead of $L_{X/k}$. Contrary to what is common in algebraic geometry, we have used homological grading. Finally we will denote by $\mathrm{QCoh}(X)$ the ∞ -category of quasi-coherent complexes constructed by Lurie in [10]. The reason for employing this category instead of the derived category is that at certain points it is important to know why things are homotopic, and not only that they are homotopic. It also allows to carry out proofs as if one was only dealing with modules, and not with complexes. Recall that a *cofiber sequence* in $\mathrm{QCoh}(X)$ consists of a sequence of morphisms $E \xrightarrow{f} F \xrightarrow{g} G$, a 2-simplex identifying the composition fg with a morphism $E \xrightarrow{h} G$, and a nullhomotopy of h . The ∞ -category $\mathrm{QCoh}(X)$ is equipped with a standard t -structure. We will use that the notion of Tor-amplitude behaves well with respect to this t -structure, i.e., if an object $E \in \mathrm{QCoh}(X)$ is of Tor-amplitude $\leq n$, then $E[m]$ is of Tor-amplitude $\leq n + m$. All details can be found in Lurie’s volumes [11, 10].

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2. REMOVING FACTORS

We first introduce the geometric objects we wish to study. These are in general Artin stacks with a fixed 1-perfect obstruction theory. The terminology *virtually smooth* for a pair of an Artin stack together with a fixed perfect obstruction theory was introduced by Fantechi–Göttsche in [5].

Definition 2.1. A pair $(X, \phi: E \rightarrow L_X)$ is a *virtually smooth Artin stack* if X is an Artin stack locally of finite type over k , E is a perfect complex of Tor-amplitude ≤ 1 , and $\text{cofib}(\phi) \in \text{QCoh}(X)_{\geq 2}$. The morphism ϕ will be referred to as the *obstruction theory*.

If X is a Deligne–Mumford stack, the morphism $\phi: E \rightarrow L_X$ in the above definition is a 1-perfect obstruction theory in the sense of Behrend–Fantechi [2]. We next define morphisms between such objects.

Definition 2.2. A *morphism of virtually smooth Artin stacks* is a pair

$$(f, \alpha): (X, \phi: E \rightarrow L_X) \longrightarrow (Y, \chi: F \rightarrow L_Y)$$

where $f: X \rightarrow Y$ is a morphism of Artin stacks over k , and $\alpha: f^*F \rightarrow E$ is a morphism of perfect complexes on X such that

$$\begin{array}{ccc} f^*F & \xrightarrow{\chi} & f^*L_Y \\ \alpha \downarrow & & \downarrow \\ E & \xrightarrow{\phi} & L_X \end{array}$$

commutes in $\text{QCoh}(X)$.

Remark 2.3. Recall that commuting in $\text{QCoh}(X)$ means that we have fixed a homotopy making the diagram commutative. This added information is absolutely essential for all further computations.

We will also need the notion of virtually smooth morphism.

Definition 2.4. Let $(f, \alpha): (X, \phi: E \rightarrow L_X) \longrightarrow (Y, \chi: F \rightarrow L_Y)$ be a morphism of virtually smooth Artin stacks. Then (f, α) is a *virtually smooth morphism* if $\text{cofib}(\alpha)$ is of Tor-amplitude ≤ 1 .

Remark 2.5. Note that a priori $\text{cofib}(\alpha)$ is only of Tor-amplitude ≤ 2 .

Derived algebraic geometry provides natural examples of virtually smooth Artin stacks and morphisms between these.

Example 2.6. Recall that a derived Artin stack X^d over k is *quasi-smooth* if its cotangent complex L_{X^d} is of Tor-amplitude ≤ 1 and its underlying Artin stack $X := t_0(X^d)$ is locally of finite type over k . By the canonical inclusion $j_X: X \hookrightarrow X^d$ we can obtain the structure of a virtually smooth Artin stack on X . The perfect obstruction theory is given by the canonical morphism $\phi: j_X^*L_{X^d} \rightarrow L_X$. Using the functoriality properties of cotangent complexes, every morphism of quasi-smooth derived Artin stacks gives rise to a morphism of virtually smooth Artin stacks.

To remove factors from obstruction theories we will make use of virtually smooth Artin stacks with some peculiar properties. We will be using virtually smooth Artin stacks $(Y, \chi: F \rightarrow L_Y)$ where the underlying Artin stack Y itself is already smooth. This is contrary to the philosophy that spaces become smooth after deriving them, or that 1-perfect obstruction theories are only interesting on very singular spaces. On the contrary, we will have to find non-smooth derived versions of spaces that are smooth, or 1-perfect obstruction theories on smooth spaces.

Definition 2.7. Let $(f, \alpha): (X, \phi: E \rightarrow L_X) \rightarrow (Y, \chi: F \rightarrow L_Y)$ be a morphism of virtually smooth schemes. We say that (f, α) is a *reduction morphism* if Y is smooth.

Remark 2.8. For applications to virtual classes, X will be assumed to be a Deligne–Mumford stack.

Given a reduction map, we would like to define a new structure of virtually smooth Artin stack on X such that the virtual dimension of X increases. The factor we would like to remove from the obstruction theory E is the pull-back to X of the fiber of $\chi: F \rightarrow L_Y$. In the following we will show that this is possible if the reduction morphism is virtually smooth. The key to removing a factor is the following lemma, which is true in much greater generality than we actually need. Note that we do not assume (f, α) either to be a reduction morphism or virtually smooth.

Lemma 2.9. Let $(f, \alpha): (X, \phi: E \rightarrow L_X) \rightarrow (Y, \chi: F \rightarrow L_Y)$ be a morphism of virtually smooth schemes. Let $K = \text{fib}(\chi)$, and define β to be the composition

$$f^*K \xrightarrow{\gamma} f^*F \xrightarrow{\alpha} E.$$

Then the composition

$$f^*K \xrightarrow{\beta} E \xrightarrow{\phi} L_X$$

is zero.

Proof. By definition, we have a cofiber sequence $K \rightarrow F \rightarrow L_Y$, and this remains a cofiber sequence after pulling to X . We thus have the following commutative diagram on X

$$\begin{array}{ccc} f^*K & \longrightarrow & 0 \\ \gamma \downarrow & & \downarrow \\ f^*F & \xrightarrow{\chi} & f^*L_Y \\ \alpha \downarrow & & \downarrow \\ E & \xrightarrow{\phi} & L_X \end{array}$$

which gives a homotopy from $\phi \circ \beta$ to zero. \square

We can now define our candidate for a reduced obstruction theory. We let $E' := \text{cofib}(\beta)$. By Lemma 2.9, we have a well-defined morphism $\phi': E' \rightarrow L_X$. Note that if we only knew the composition to be zero in the derived category this would not be sufficient to obtain a well-defined morphism.

Theorem 2.10. *Let $(f, \alpha): (X, \phi: E \rightarrow L_X) \rightarrow (Y, \chi: F \rightarrow L_Y)$ be a virtually smooth reduction map. Then*

$$(X, \phi': E' \rightarrow L_X)$$

is a virtually smooth Deligne–Mumford stack.

Proof. We first show that E' is perfect. Let K as above denote $\text{fib}(\chi)$, so that we have a cofiber sequence

$$K \rightarrow F \xrightarrow{\chi} L_Y.$$

Now F is perfect by assumption, and L_Y is perfect since Y is smooth and locally of finite presentation. Since the property of being perfect is stable under cofiber sequences, K is perfect, and thus f^*K is perfect. This shows that E' is the cofiber of a morphism between perfect objects, and thus is perfect.

We now want to show that E' is of Tor-amplitude ≤ 1 . Since Y is smooth and L_Y thus is of Tor-amplitude ≤ 0 , the above cofiber sequence shows that K is of Tor-amplitude ≤ 1 . It follows that f^*K is also of Tor-amplitude ≤ 1 . Let γ denote the morphism $f^*K \rightarrow f^*F$. By definition, the diagram

$$\begin{array}{ccc} f^*K & \xrightarrow{\gamma} & f^*F \\ & \searrow \beta & \downarrow \alpha \\ & & E \end{array}$$

commutes. This gives us a cofiber sequence $\text{cofib}(\gamma) \rightarrow \text{cofib}(\beta) \rightarrow \text{cofib}(\alpha)$. Since $E' = \text{cofib}(\beta)$ and $f^*L_Y = \text{cofib}(\gamma)$, we have a cofiber sequence

$$f^*L_Y \rightarrow E' \rightarrow \text{cofib}(\alpha).$$

Since we assumed (f, α) a virtually smooth morphism, $\text{cofib}(\alpha)$ is of Tor-dimension ≤ 1 . Again using that Y is smooth, it follows that E' is of Tor-dimension ≤ 1 .

It remains to show that $\text{cofib}(\phi') \in \text{QCoh}(X)_{\geq 2}$, or equivalently that $\text{fib}(\phi') \in \text{QCoh}(X)_{\geq 1}$. Let $K' = \text{fib}(\phi)$. Since the composition $\phi \circ \beta$ factors over zero, we obtain a morphism $\delta: f^*K \rightarrow K'$, and we can identify $\text{fib}(\phi')$ with $\text{cofib}(\delta)$. Since f^*K and K' are both in $\text{QCoh}(X)_{\geq 1}$, the claim follows. \square

Remark 2.11. Since $(X, \phi': E' \rightarrow L_X)$ is a virtually smooth Artin stack this automatically poses the question if this obstruction theory is induced by a natural structure of a derived Artin stack on X . Adding plenty of assumptions such a statement indeed holds. First of all, we have to assume that the perfect obstruction theories $(E \rightarrow L_X)$ and $(F \rightarrow L_Y)$ are induced by derived stacks X^d and Y^d , and the compatibility datum α is induced by a morphism $f^d: X^d \rightarrow Y^d$. Furthermore, we have to assume that the derived structure on Y^d splits as

$$Y^d = Y \times Y^{\text{der}}.$$

The underling stack of Y^{der} is a point. Let $p: Y^d \rightarrow Y^{\text{der}}$ be the projection. The homotopy fiber product of the diagram

$$\begin{array}{ccc} & X^d & \\ & \downarrow p \circ f^d & \\ \text{Spec } k & \longrightarrow & Y^{\text{der}} \end{array}$$

then yields the desired derived Artin stack. Such a splitting exists for the derived Picard stack of a $K3$ -surface. It is reasonable to expect such a splitting to exist whenever Y^d is a group stack with smooth truncation.

3. APPLICATION TO DEFORMATION THEORY

In the following assume that $(X, \phi: E \rightarrow L_X)$ is a virtually smooth Deligne–Mumford stack. Given a reduction map $(f, \alpha): (X, \phi: E \rightarrow L_X) \rightarrow (Y, \chi: F \rightarrow L_Y)$, we can define a *generalized semi-regularity map*. Given a morphism $p: T \rightarrow X$ where $T = \text{Spec}(A)$ is an affine scheme, and a square-zero extension $T \hookrightarrow T'$ classified by a morphism $\eta: L_T \rightarrow M[1]$ for some A -module M , $p: T \rightarrow X$ extends to a morphism $T' \rightarrow X$ if and only if the element in $\text{Ext}^1(p^*E, M)$ defined by the homotopy class of the composition

$$p^*E \xrightarrow{p^*\phi} p^*L_X \longrightarrow L_T \xrightarrow{\eta} M[1]$$

vanishes. We define the generalized semi-regularity map to be the map

$$\text{Ext}^1(p^*E, M) \longrightarrow \text{Ext}^1(p^*f^*F, M)$$

obtained by composition with α . We will now show that realized obstructions lie in the kernel of the generalized semi-regularity map. We first give a definition of realized obstructions following Behrend–Fantechi [2].

Definition 3.1. Let $(X, \phi: E \rightarrow L_X)$ be a virtually smooth Deligne–Mumford stack, and let $p: T \rightarrow X$ be a morphism with $T = \text{Spec}(A)$. Let M be a A -module. A non-zero morphism $\alpha: p^*E \rightarrow M[1]$ *realizes an obstruction* if there exists a square-zero extension $T \hookrightarrow T'$ classified by $\eta: L_T \rightarrow M[1]$ such that

$$\begin{array}{ccccc} p^*E & \longrightarrow & p^*L_X & \longrightarrow & L_T \\ & \searrow \alpha & & & \downarrow \eta \\ & & & & M[1] \end{array}$$

commutes.

We can now show that obstructions that are actually realized always lie in the kernel of the generalized semi-regularity map.

Proposition 3.2. *Let $(f, \alpha): (X, \phi: E \rightarrow L_X) \rightarrow (Y, \chi: F \rightarrow L_Y)$ be a reduction morphism. Assume that X is a Deligne–Mumford stack, and let $p: T \rightarrow X$ be a morphism where T is an affine scheme. Then realized obstructions lie in the kernel of the generalized semi-regularity map.*

Proof. Since Y is smooth, this allow us to conclude that $\text{Ext}^1(p^*f^*L_Y, M)$ and $\text{Ext}^2(p^*f^*L_Y, M)$ are zero. Using the pull-back of the cofiber sequence

$$K \longrightarrow F \xrightarrow{\chi} L_Y$$

to T we thus have an isomorphism

$$\mathrm{Ext}^1(p^*f^*F, M) \simeq \mathrm{Ext}^1(p^*f^*K, M).$$

Applying Lemma 2.9 the claim follows. \square

Remark 3.3. A reduction morphism is virtually smooth if its generalized semi-regularity morphism is surjective.

4. APPLICATION TO MODULI OF MAPS

We now apply the formalism developed above in an example, working over $k = \mathbb{C}$. The example we will be concerned with is the moduli space of maps from a fixed curve C to a smooth projective complex surface S satisfying the condition $c_1(\mathbb{R}f_*\mathcal{O}_C) = \beta$, where $\beta \in H^1(S, \Omega_S^1)$. We will denote this space by $\mathrm{Mor}_\beta(C, S)$. It is well-known that this space is actually a virtually smooth scheme, see Behrend–Fantechi [2]. We will denote this space equipped with its standard obstruction theory by

$$\left(\mathrm{Mor}_\beta(C, S), \phi: E \rightarrow L_{\mathrm{Mor}_\beta(C, S)} \right).$$

To apply the results of [17], it is important to note that the same structure of virtually smooth scheme can also be constructed using Example 2.6. To see this, denote by $i: \mathrm{St}_k \rightarrow \mathrm{dSt}_k$ the inclusion functor from stacks over k to derived stacks over k . We can then define the derived moduli space of maps to be the derived scheme parametrizing morphisms in this larger category. We will denote this derived scheme by $\mathbb{R}\mathrm{Mor}_\beta(C, S)$.

In order to remove a factor from the obstruction theory using the above formalism we have to find some virtually smooth Artin stack as comparison space. The natural candidate in this example is the Picard stack $\mathrm{Pic}(S) := \underline{\mathrm{Hom}}_{\mathrm{St}_k}(S, B\mathbb{G}_m)$. As above, there again is a derived version of this stack, given by $\mathbb{R}\mathrm{Pic}(S) := \underline{\mathrm{Hom}}_{\mathrm{dSt}_k}(S, B\mathbb{G}_m)$. Denote the canonical inclusion by $j: \mathrm{Pic}(S) \rightarrow \mathbb{R}\mathrm{Pic}(S)$. The virtually smooth space we will use as target for our potential reduction morphism is

$$\left(\mathrm{Pic}(S), \chi: j^*L_{\mathbb{R}\mathrm{Pic}(S)} \rightarrow L_{\mathrm{Pic}(S)} \right).$$

Since the underlying Artin stack $\mathrm{Pic}(S)$ is smooth, this is an excellent candidate for a reduction map.

Finally, we have to give a map of virtually smooth schemes. In [17], a map

$$\mathbb{R}\mathrm{Mor}_\beta(C, S) \xrightarrow{A_S} \mathbb{R}\mathrm{Perf}(S) \xrightarrow{\det} \mathbb{R}\mathrm{Pic}(S)$$

is given. Using Example 2.6, we obtain a map of virtually smooth schemes

$$(f, \alpha): \left(\mathrm{Mor}_\beta(C, S), \phi: E \rightarrow L_{\mathrm{Mor}_\beta(C, S)} \right) \longrightarrow \left(\mathrm{Pic}(S), \chi: j^*L_{\mathbb{R}\mathrm{Pic}(S)} \rightarrow L_{\mathrm{Pic}(S)} \right).$$

Remark 4.1. The generalized semi-regularity map associated to (f, α) is just the semi-regularity map for morphisms of Buchweitz–Flenner [4, Remark 7.24].

We can now define a new structure of virtually smooth scheme on $\mathrm{Mor}_\beta(C, S)$. Let $K := \mathrm{fib}(\chi)$. Note that K is non-trivial if and only if $H^2(S, \mathcal{O}_S)$ is non-trivial. As above, we then have a morphism $\gamma: f^*K \rightarrow E$, and can define $E' := \mathrm{cofib}(\gamma)$ as candidate for a reduced obstruction theory.

Corollary 4.2. *Assume that*

$$(f, \alpha): \left(\mathrm{Mor}_\beta(C, S), \phi: E \rightarrow L_{\mathrm{Mor}_\beta(C, S)} \right) \longrightarrow \left(\mathrm{Pic}(S), \chi: j^* L_{\mathbb{R}\mathrm{Pic}(S)} \rightarrow L_{\mathrm{Pic}(S)} \right).$$

is virtually smooth. Then

$$\left(\mathrm{Mor}_\beta(C, S), \phi': E' \rightarrow L_{\mathrm{Mor}_\beta(C, S)} \right)$$

is a virtually smooth scheme.

Remark 4.3. Note that the two structures of virtually smooth scheme on $\mathrm{Mor}_\beta(C, S)$ only differ in case $H^2(S, \mathcal{O}_S)$ is non-zero.

Example 4.4. Assume that S is a K3-surface. Then the morphism (f, α) is virtually smooth for any class $\beta \neq 0$.

We finally want to state a condition ensuring that (f, α) is virtually smooth. This condition was identified by Kool and Thomas [9] and provided the motivation for this work.

Proposition 4.5. *Assume that*

$$H^1(S, T_S) \xrightarrow{\cup \beta} H^2(S, \mathcal{O}_S)$$

is surjective. Then (f, α) is virtually smooth.

Proof. It suffices to prove the statement on k -points. We thus have to show that at any point $p: \mathrm{Spec} k \rightarrow \mathrm{Mor}_\beta(C, S)$ the morphism

$$\pi_1(p^* \alpha): \pi_1(p^* L_{\mathbb{R}\mathrm{Pic}(S)}) \longrightarrow \pi_1(p^* E)$$

is injective. Equivalently, we have to show that the dual of $\pi_1(p^* \alpha)$ is surjective.

Let $g: C \rightarrow S$ be the morphism corresponding to p . Recall from Illusie [8, Chapitre V] or explicitly from [6] that for any perfect complex the first Chern class factors as composition of the Atiyah class and the trace-map. Let $E = \mathbb{R}g_* \mathcal{O}_C$. Thus, for any class $\alpha \in H^1(S, T_S)$ the operation of cup-product with $\beta = c_1(E)$ factors as

$$\begin{array}{ccc} H^1(S, T_S) & & \\ \downarrow -\cup \mathrm{at}_E & \searrow -\cup \beta & \\ \mathrm{Ext}_S^2(E, E) & \xrightarrow{\mathrm{tr}} & H^2(S, \mathcal{O}_S). \end{array}$$

Now in [17, Appendix] it is shown that $-\cup \mathrm{at}_E$ factors as

$$\begin{array}{ccc} & & H^1(S, T_S) \\ & \swarrow & \downarrow -\cup \mathrm{at}_E \\ H^1(C, g^* T_S) & \xrightarrow{T_{A_S, g}} & \mathrm{Ext}_S^2(E, E). \end{array}$$

Here $T_{A_S, g}$ is the tangent to

$$A_S: \mathbb{R}\mathrm{Mor}_\beta(C, S) \longrightarrow \mathbb{R}\mathrm{Perf}(S)$$

at the point p . Piecing the two diagrams together, we arrive at a commutative diagram

$$\begin{array}{ccccc} & H^1(S, T_S) & & & \\ & \swarrow \scriptstyle -\cup \mathrm{at}_E \quad \downarrow & \searrow \scriptstyle -\cup \beta & & \\ H^1(C, g^*T_S) & \xrightarrow{T_{A_S}} & \mathrm{Ext}_S^2(E, E) & \xrightarrow{\mathrm{tr}} & H^2(S, \mathcal{O}_S). \end{array}$$

Since the bottom row is the dual of $\pi_1(p^*\alpha)$ and by assumption

$$H^1(S, T_S) \xrightarrow{\cup \beta} H^2(S, \mathcal{O}_S)$$

is surjective, the claim follows. \square

Remark 4.6. Behrend and Fantechi in [3] suggested removing a factor of $H^0(X, \Omega_X^2)$ from the obstruction theory of the moduli space of stable maps to an irreducible complex symplectic variety of dimension n to perform refined curve counts. The formalism developed here applies as soon as one has an appropriate target for a reduction morphism. Promising candidates are the derived version of the intermediate Jacobian J_X^p with $p = n - 1$ constructed recently by Pridham [16] and Iacono–Manetti [7]. More generally, this should work for any variety for which an analogue of the surjectivity of cup-product with β holds.

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